Finite Math - Spring 2017

Lecture Notes - 4/5/2017

Homework

- Section 4.4 - 55, 57, 58
- Section 4.5 - 19, 20, 25, 26, 29, 32, 33, 35, 39, 42, 45, 51, 55, 61, 64

Section 4.4 - Matrices: Basic Operations

Example 1. *Find a, b, c, and d such that*

$$\begin{bmatrix} 6 & -5 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -16 & 64 \\ 24 & -6 \end{bmatrix}$$

Solution. *If we multiply out the matrices on the left, we get the equation*

$$\begin{bmatrix} 6a - 5c & 6b - 5d \\ 3c & 3d \end{bmatrix} = \begin{bmatrix} -16 & 64 \\ 24 & -6 \end{bmatrix}$$

And so we have the following two systems of equations

$$\begin{aligned} 6a - 5c &= -16 \\ 3c &= 24 \end{aligned}$$

and

$$\begin{aligned} 6b - 5d &= 64 \\ 3d &= -6 \end{aligned}$$

The augmented matrices for these two systems are

$$\left[\begin{array}{cc:c} 6 & -5 & -16 \\ 0 & 3 & 24 \end{array}\right] \qquad \textit{and} \qquad \left[\begin{array}{cc:c} 6 & -5 & 64 \\ 0 & 3 & -6 \end{array}\right]$$

Notice that these two systems have the same coefficient matrix! In the first augmented matrix, we are aiming to solve for the variables a and c which are the first column of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and in the second augmented matrix, we are solving for the variables b and d. Because of this, we can just stick the augmented matrices together to form the new one

$$\left[\begin{array}{cc:cc} 6 & -5 & -16 & 64 \\ 0 & 3 & 24 & -6 \end{array}\right]$$

Then, as before, we aim to get a reduced form on the left side, and that will simultaneously solve both systems. That is, we are aiming for

$$\left[\begin{array}{cc:cc} 1 & 0 & a & b \\ 0 & 1 & c & d \end{array}\right]$$

So, let's solve the system

$$\left[\begin{array}{cc:cc} 6 & -5 & -16 & 64 \\ 0 & 3 & 24 & -6 \end{array}\right] \overset{\frac{1}{6}R_1 \to R_1}{\sim} \left[\begin{array}{cc:cc} 1 & -\frac{5}{6} & -\frac{16}{6} & \frac{64}{6} \\ 0 & 3 & 24 & -6 \end{array}\right] \overset{\frac{1}{3}R_2 \to R_2}{\sim} \left[\begin{array}{cc:cc} 1 & -\frac{5}{6} & -\frac{16}{6} & \frac{64}{6} \\ 0 & 1 & 8 & -2 \end{array}\right]$$

$$\overset{R_1 + \frac{5}{6}R_2 \to R_1}{\sim} \left[\begin{array}{cc:cc} 1 & 0 & 4 & 9 \\ 0 & 1 & 8 & -2 \end{array}\right]$$

So, we get that $a = 4$, $b = 9$, $c = 8$, *and* $d = -2$.

Example 2. *Find* a, b, c, *and* d *such that*

$$\begin{bmatrix} 6 & -5 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{5}{18} \\ 0 & \frac{1}{3} \end{bmatrix}$$

Section 4.5 - Inverse of a Square Matrix

The Identity Matrix.

Example 3. *Find the products*

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ *and* $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ *and* $\begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution.

(a)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1+0 & 2+0 \\ 0+3 & 0+4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+2 \\ 3+0 & 0+4 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 2+0 & 4+0 & 6+0 \\ 0+3 & 0+6 & 0+9 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2+0+0 & 0+4+0 & 0+0+6 \\ 3+0+0 & 0+6+0 & 0+0+9 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Definition 1 (Identity Matrix)**.** *The* $n \times n$ identity matrix *is a matrix, denoted by* I *or* I_n*, which is an* $n \times n$ *matrix with* 1*'s on the primcipal diagonal and* 0*'s everywhere else. For an* $m \times n$ *matrix* M*, we have*

$$I_m M = M = M I_n.$$

From the above example, we have the two forms of the identity matrix

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse of a Square Matrix. It is only possible to find a multiplicative inverse of a matrix if it is a square matrix. So, we now restrict ourselves to square matrices.

Definition 2 (Inverse Matrix). *If M is a square matrix of size n, and if there is a matrix, denoted M^{-1}, such that*

$$MM^{-1} = M^{-1}M = I_n$$

we call M^{-1} the inverse *of M. If M does not have an inverse, then M is called a* singular matrix*.*

Example 4. *Find the inverse of the matrix*

$$M = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}.$$

Solution. *Since we're looking for a matrix which satisfies $MX = I_n$, we can use the trick from the previous section and just row reduce $[M|A]$ to the form $[I_n|X]$, then check that $XM = I_n$ as well.*

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{array}\right] \overset{R_2 - 2R_1 \to R_2}{\sim} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right]$$

$$\overset{-R_2 \to R_2}{\sim} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right] \overset{R_1 - 2R_2 \to R_1}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

So we should have that $M^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. We need to check that $M^{-1}M = I_2$.

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4-3 & 6-6 \\ -2+2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Thus we have that

$$M^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}.$$

Example 5. *Find the inverse of the matrix*

$$N = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

Solution.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \overset{R_2 - 2R_1 \to R_2}{\sim} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

Since there is only zeros in the bottom line of the left side, the matrix N does not have an inverse.

Example 6. *Find the inverse of the matrix*

$$M = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 2 & -3 \\ -2 & -3 & -1 \end{bmatrix}.$$

Solution.

$$\left[\begin{array}{ccc|ccc} 2 & 2 & 0 & 1 & 0 & 0 \\ 1 & 2 & -3 & 0 & 1 & 0 \\ -2 & -3 & -1 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 & 0 & 0 \\ -2 & -3 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{R_3 + R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 1 \end{array}\right] \underset{\sim}{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & -2 & 6 & 1 & -2 & 0 \\ 0 & -1 & -1 & 1 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{R_2 - 3R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & 1 & 9 & -2 & -2 & -3 \\ 0 & -1 & -1 & 1 & 0 & 1 \end{array}\right] \underset{\sim}{R_3 + R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 0 & 1 & 0 \\ 0 & 1 & 9 & -2 & -2 & -3 \\ 0 & 0 & 8 & -1 & -2 & -2 \end{array}\right]$$

$$\underset{\sim}{R_1 - 2R_2 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -21 & 4 & 5 & 6 \\ 0 & 1 & 9 & -2 & -2 & -3 \\ 0 & 0 & 8 & -1 & -2 & -2 \end{array}\right] \underset{\sim}{\tfrac{1}{8}R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -21 & 4 & 5 & 6 \\ 0 & 1 & 9 & -2 & -2 & -3 \\ 0 & 0 & 1 & -\frac{1}{8} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$$

$$\underset{\sim}{R_1 + 21R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{11}{8} & -\frac{1}{4} & \frac{3}{4} \\ 0 & 1 & 9 & -2 & -2 & -3 \\ 0 & 0 & 1 & -\frac{1}{8} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right] \underset{\sim}{R_1 + 21R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{11}{8} & -\frac{1}{4} & \frac{3}{4} \\ 0 & 1 & 1 & -\frac{7}{8} & \frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & -\frac{1}{8} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right]$$

Theorem 1. *To find the inverse of an $n \times n$ matrix M, one begins with the augmented matrix $[M|\, I_n]$ and uses row operations to transform it into $\left[I_n|\, M^{-1}\right]$. However, if one or more rows of all 0's appear on the left side of the augmented matrix, M is not invertible, i.e., M^{-1} does not exist.*

Example 7. *Find the inverse of the following matrices (if possible):*

(a)

$$M = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$$

(b)

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

(c)

$$N = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$$

Remark 1. *There is a trick to invert a* 2×2 *matrix. If* $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *then*

$$M^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$